Derived Pushforward

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13.12.24 Lecture ongoing

$f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a map of ringed spaces.

The right derived functors of $f_* : \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_Y}$ measure the failure of right exactness of f_*.

<u>Prop:</u> f as above; let $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$ be a flasque sheaf. Then $\mathcal{F}$ is f_* acyclic; i.e

$$R^i f_* \mathcal{F} = 0 \quad \forall i > 0$$

<u>Pf.</u> Take an injective resolution

$$0 \to \mathcal{F} \to I^\bullet \quad \text{in } \mathrm{Mod}_{\mathcal{O}_X}$$

Since $R^i f_* \mathcal{F} := H^i(f_* I^\bullet)$, it is enough to show that $H^i(f_* I^\bullet) = 0 \ \forall i > 0$; or that the complex $f_* I^\bullet$ in $\mathrm{Mod}_{\mathcal{O}_Y}$ is exact at i-th spot for $i \geq 1$.

For any open $V \subseteq Y$,

$$\begin{array}{ccccccc}
0 \to f_* I^0(V) & \to & f_* I^1(V) & \to \cdots & \to & f_* I^j(V) & \to \cdots \\
\| & & \| & & & \| & \\
I^0(f^{-1}(V)) & \to & I^1(f^{-1}(V)) & \to \cdots & \to & I^j(f^{-1}(V)) & \to \\
\| & & & & & \| & \\
\Gamma(f^{-1}(V), I^0|_{f^{-1}(V)}) & \to & \cdots & & \to & \Gamma(f^{-1}(V), I^j|_{f^{-1}(V)}) &
\end{array}$$

But $\mathcal{F}|_{f^{-1}(V)} \to I^\bullet|_{f^{-1}(V)}$ is a flasque resolution and

$\mathcal{F}|_{f^{-1}(V)}$ is flasque. So $H^i(f_* I^\bullet(V)) = 0 \quad \forall i > 0, \forall V$

This implies that the complex of stalks

$$(f_* I^\bullet)_y \text{ is exact at degree } \geq 1.$$

<u>Thm:</u> Let $f : X \to Y$ be a morphism of schemes where X is noetherian. For a quasi-coherent sheaf $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$, $R^i f_* \mathcal{F}$ is quasi-coherent $\forall i$.

Pf. $i=0$ has already been established.

$\mathcal{F}$ admits a resolution by q.coh flasque sheaves

$$\mathcal{F} \to \mathcal{J}^\bullet \quad \text{, as } X \text{ is noeth.}$$

Since flasque sheaves are f_* acyclic

$$R^i f_* \mathcal{F} = H^i(f_* \mathcal{J}^\bullet)$$

Since $Qcoh(Y)$ is abelian and each $f_* \mathcal{J}^\alpha$ is q.coh, $H^i(f_* \mathcal{J}^\bullet)$ is q.coh.

Proof: Given $f: X \to Spec(A)$, where X noetherian and $\mathcal{F} \in Qcoh(X)$, $R^i f_* \mathcal{F} \xrightarrow{\sim} H^i(X, \mathcal{F})^\sim$

[Note $H^i(X, \mathcal{F})$ are A-mods]

Pf. Take a res of $\mathcal{F}$ by q.coh flasque sheaves

$$\mathcal{F} \to \mathcal{J}^\bullet$$

Since $f_* \mathcal{J}^\bullet$ is a complex of q.coh sheaves on $Spec(A)$, and $M \to \tilde{M}$ is an exact functor,

$$H^i(f_* \mathcal{J}^\bullet)(Spec(A)) = H^i(\Gamma(Spec(A), -) f_* \mathcal{J}^\bullet)$$

$$= H^i\left(\mathcal{J}^0(X) \to \mathcal{J}^1(X) \to \cdots\right)$$

$$= H^i(X, \mathcal{F})$$

Since $R^i f_* \mathcal{F} = H^i(f_* \mathcal{J}^\bullet)$ is q.coh

$$R^i f_* \mathcal{F} \simeq H^i(f_* \mathcal{J}^\bullet)(Spec(A))^\sim$$

Proof: Let $f: X \to Y$ be a morph of schemes, X be noetherian. $R^i f_* \mathcal{F} \in Mod_{\mathcal{O}_Y}$ can be thought of as the quasi coherent sheaf whose value on an affine open $V \subseteq Y$ is $H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$.

Pf. Follows from the observation

$$R^i f_* \mathcal{F}|_V = R^i (f|_{f^{-1}(V)})_* \mathcal{F}|_{f^{-1}(V)} .$$

Thm: Let $f: X \to Y$ be a scheme map.

① For $\mathcal{F} \in \mathrm{Mod}\, \mathcal{O}_X$, if $R^i f_* \mathcal{F} = 0 \quad \forall\ i > 0$,
Then

$$H^i(X, \mathcal{F}) \xrightarrow{\sim} H^i(Y, f_* \mathcal{F}) \qquad \forall i$$

Assume that X is noeth.

② If f is an affine morphism, i.e. inverse image of any affine open is an affine open under f.

$$R^i f_* \mathcal{F} = 0 \qquad \forall\ \mathcal{F} \in \mathrm{Qcoh}(X),\ i > 0.$$

③ Assume f is finite / closed immersion, same assertion as ②

Pf. Given an injective resolution

$$0 \to \mathcal{F} \to I^\bullet$$

$$f_* \mathcal{F} \to f_* I^\bullet \quad \text{is also exact by our hypothesis}$$

Each $f_* I^j$ is flasque, so $f_* I^\bullet$ is a flasque resolution of $f_* \mathcal{F}$. Then $H^i(Y, f_* \mathcal{F})$

$$\simeq H^i\left(\Gamma(Y, -)(f_* I^\bullet)\right)$$

$$\simeq H^i\left(I^0(X) \to I^1(X) \to \cdots \right)$$

$$\simeq H^i(X, \mathcal{F})$$

② Since $R^i f_* \mathcal{F}$ is q.coh, it is enough to show that given an affine open $V \subseteq Y$ and $i > 0$,

$$R^i f_* \mathcal{F}(V) = 0$$

But $R^i f_* \mathcal{F}(V) \xrightarrow{\sim} H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$

Since $f^{-1}(V)$ is affine and $\mathcal{F}|_{f^{-1}(V)}$ is q.coh, we are done.

(2) Both classes of morphisms are affine.

<u>Rmk.</u> Given a closed immersion $i : Z \hookrightarrow X$, $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$, one often denotes

$$H^i(X, i_* \mathcal{F}) \text{ by } H^i(X, \mathcal{F}).$$